Monte-Carlo Pricing under a Hybrid Local Volatility model

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Key Interests in Finance

- Pricing of exotic derivatives
- Monte-Carlo simulations
- Local Volatility model for Foreign Exchange Rates (FX)
- Hybrid with Interest Rate models (IR)

Introduction

Key Interests in Finance

- Pricing of exotic derivatives
- Monte-Carlo simulations
- Local Volatility model for Foreign Exchange Rates (FX)
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Key Interests in CUDA

- High-dimensional Monte-Carlo simulations
- Texture memory (layered)

Plan of the talk

- Description of the problem and motivation for parallel programming and textures

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- Numerical tests
  - Call/Put options in Local Volatility (LV) model
  - Exotic swaps in LV model
  - Exotic swaps in Hybrid LV model

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- Description of the problem and motivation for parallel programming and textures
- Outline of implementation in CUDA
- Numerical tests
  - Call/Put options in Local Volatility (LV) model
  - Exotic swaps in LV model
  - Exotic swaps in Hybrid LV model
- Conclusion on performance and use in industry

Description of the problem

The product: Power-Reverse Dual Coupon Swap (PRDC)

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  - receive option on FX_i with strike K_i:

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  - pay option on IR_i with strike Q_i:

$$-max(IR_i - Q_i, 0)$$

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  - if $C_i > A$, cancel all remaining cash-flows

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The product: Main features

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- Sensitive to FX smile
  $\rightarrow$ modelling of smile
- Sensitive to FX-IR correlation, IR volatility
  $\rightarrow$ modelling of IR stochasticity
  $\rightarrow$ multi-factor FX-IR hybrid
- Path-dependent due to exotic exercise
  $\rightarrow$ mainly Monte-Carlo

Description of the problem

The model: Dupire's Local Volatility [1]

- Diffusion with volatility $\sigma(t, FX)$

$$\frac{dFX}{FX} = (r_d - r_f)dt + \sigma(t, FX)dW$$

  - r_d is the domestic interest rate
  - r_f is the foreign interest rate
  - dW is a Brownian motion

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- Model fits the market *exactly* for Dupire's condition

$$\sigma^2(t, FX) = f\left(\frac{\partial\theta}{\partial t}, \frac{\partial\theta}{\partial FX}, \frac{\partial^2\theta}{\partial FX^2}\right)$$

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  - $\rightarrow$ texture memory [2]
  - $\rightarrow$ simple but lacks flexibility

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- Linear interpolation in FX at known t

  $\rightarrow$ *layered* textures

  $\rightarrow$ slightly more complicated but more flexible and/or accurate

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  - $\rightarrow$ Monte-Carlo simulation
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- Model requires interpolation of a matrix
  - $\rightarrow$ benefit from texture memory
- Multiple cash-flows, monitoring, smile-modelling
  - $\rightarrow$ large number of time steps
  - $\rightarrow$ high-dimensional problem
  - $\rightarrow$ inline random number generation

Implementation Outline

Single-thread:

- On each path j, at each time t_n

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- Loop on path, then time.

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  - 1-dimensional blocks of $N_{threads}$ threads
  - $s = N_{blocks} \times N_{threads}$ = number of concurrent threads

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- Thread j calculates paths $j, j+s, j+2s, etc...$
- Thread j must remember previous spot values for paths $j, j+s, j+2s, etc...$

  $\rightarrow$ too much for shared memory

  $\rightarrow$ store previous spot values in global memory

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  - calculates products on paths $j, j+s, j+2s, etc...$
  - sums them in local variable
  - writes sums in shared memory
- Synchronize
- In each block:
  - one thread is attributed to each product
  - accumulates in a local variable all thread sums for this product
  - writes "block-partial" sum in global memory

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- On host, sum results of all blocks.

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- typical number of simulations: 100K, but may want more
- global generation requires minimum global memory

$$500 \times 2 \times 100,000 \times 4 = 400MB$$

- cannot run on all devices, too restrictive for pratical applications

  $\rightarrow$ use inline generation

Texture:

- Desired interpolation

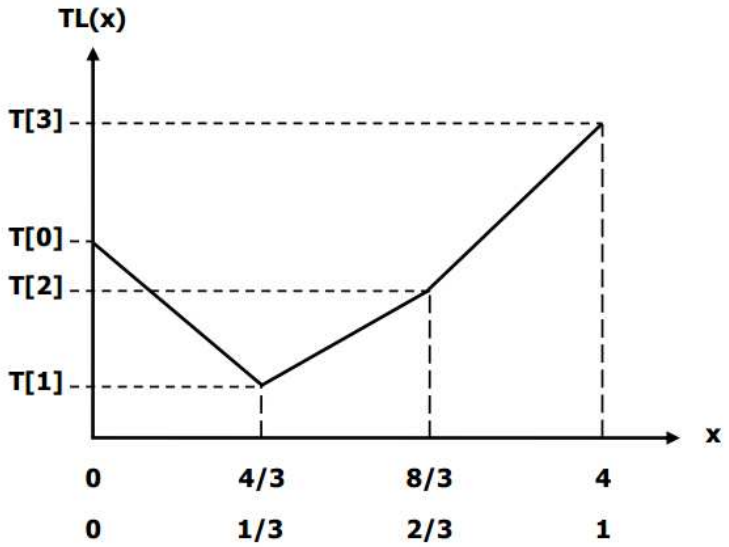

Figure E-3. One-Dimensional Table Lookup Using Linear Filtering

Texture:

- Texture interpolation

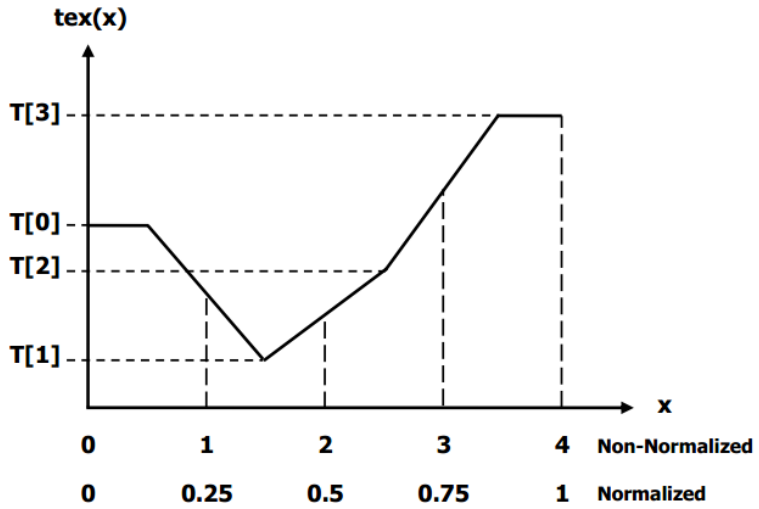

Figure E-2. Linear Filtering of a One-Dimensional Texture of Four Texels in Clamp Addressing Mode

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Texture:

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- The volatility at any spot FX is

$$\sigma(FX) = tex(\alpha FX + \beta)$$

$$\text{with} \begin{cases} \alpha = \dfrac{M-1}{M(FX_{M-1} - FX_0)} \\ \beta = \dfrac{1}{M}\left(\dfrac{1}{2} - (M-1)\dfrac{FX_0}{FX_{M-1} - FX_0}\right) \end{cases}$$

Implementation Outline

Texture:

- Bi-linear interpolation with standard texture

$$\sigma(t, FX) = \mathrm{tex2D}(\alpha FX + \beta, \gamma t + \delta)$$

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Texture:

- Bi-linear interpolation with standard texture

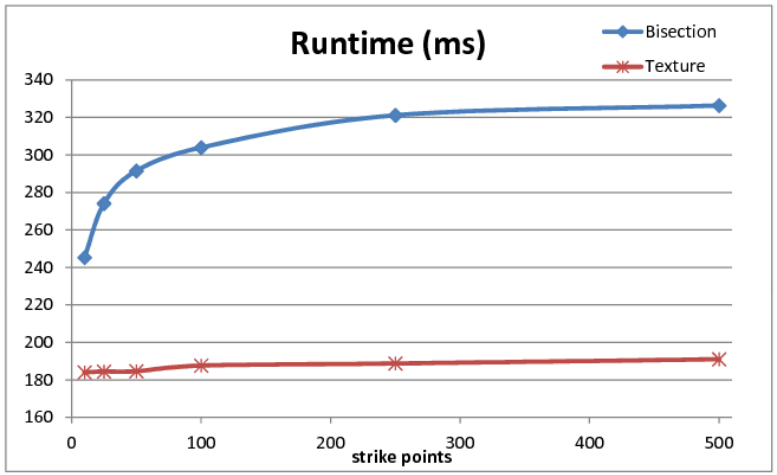
$$\sigma(t, FX) = \mathrm{tex2D}(\alpha FX + \beta, \gamma t + \delta)$$

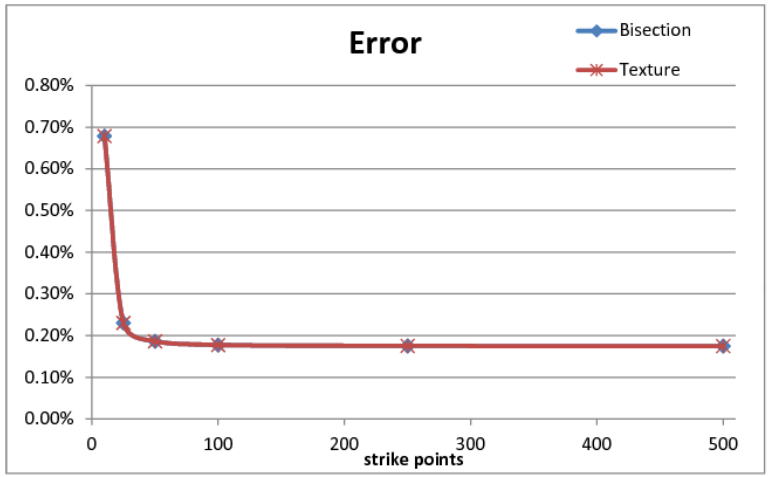
- Linear interpolation with layered texture

$$\sigma(t_n, FX) = \mathrm{tex1DLayered}(\alpha FX + \beta, n)$$

Vanilla Options:

- Performance of the texture (500 time steps, 500K simulations)

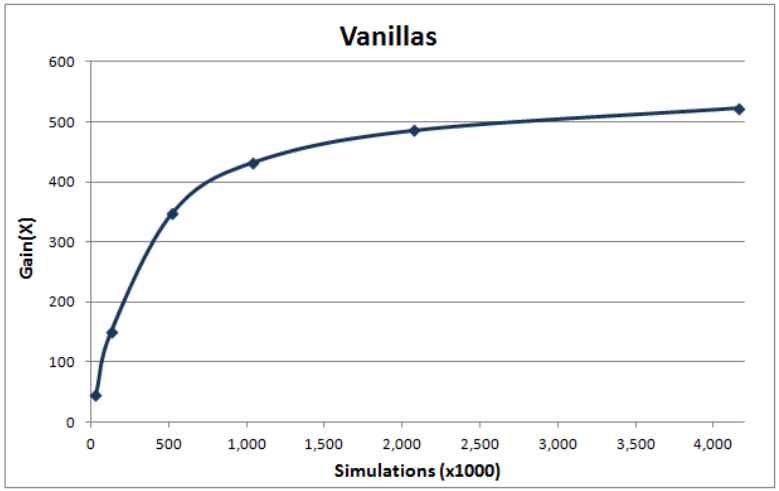




- 50% $\sim$ 70% speed gains with texture
- good accuracy of the texture interpolation
- $\sim$ 100 points sufficient

Vanilla Options:

- Gain (single thread vs. GTX 460)



Exotic Swap (one factor):

- Additional state variable on path j

$$C_i^j = \sum_{k=1}^{i} max(FX_k^j - K_k, 0)$$

$\rightarrow$ one more read/write access from global memory

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- Product calculated only at cash-flow times (at most 120)

  $\rightarrow$ less operations than for vanillas (500)

Exotic Swap (one factor):

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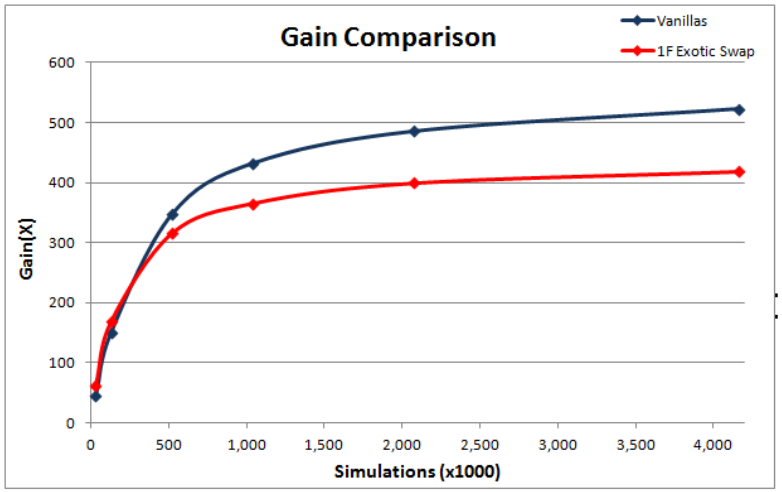

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$$dr_d = (\theta - ar_d)dt + \sigma_r dW_r$$

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- 2 additional state variable on path j

$$r^j, \quad e^{\int_0^T r^j dt} \text{ (numeraire)}$$

$\rightarrow$ two more read/write accesses to global memory

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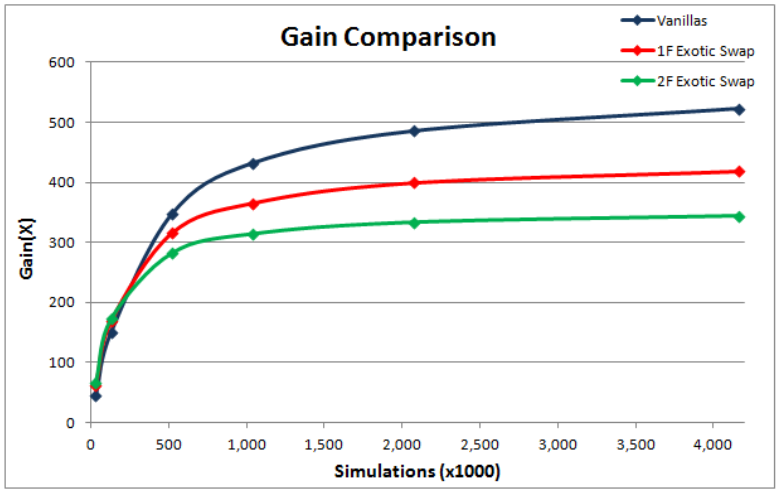

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- Extention to callables

  $\rightarrow$ Longstaff-Schwartz not entirely parallel

  $\rightarrow$ should not be a problem but gains may be lower

  $\rightarrow$ use Malliavin calculus? (Abbas-Turki, GTC 2010)

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- Value-at-Risk and Potential Exposure calculations possible without approximations
- Large number of scenario testing possible on exotic portfolios

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